OPENWIND THEORETICAL BASIS AND VALIDATION*

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0. BACKGROUND AND PURPOSE

openWind is a new software program developed by AWS Truewind, LLC, as an aid for the design, optimization, and assessment of wind power projects. It is built around an open-source platform for maximum transparency and to encourage the growth of a community of software users and developers who will keep the program at the cutting edge of technology. In its user interface, data types, and architecture, the software is patterned from Geographical Information Systems (GIS). This enables the program to be applied effectively and efficiently to a wide range of wind farm design challenges of varying complexity and size. However, its core energy computations are designed to be functionally identical to those of other leading wind farm design programs.

The purpose of this document is to describe the equations that govern openWind's energy capture and wake loss calculations. In addition, the document presents the results of validation tests carried out by AWS Truewind to confirm that the equations are sound and have been applied correctly. This validation exercise employed data from 20 real-world cases and compared the resulting energy production calculations with those from another leading wind farm design program.

In the following sections, the basic energy capture, wake effects, non-wake energy losses, and air density adjustments are described. After this, the validation procedure and results are presented, followed by a glossary of terms and references, and an appendix on computational methods.

1. ENERGY CAPTURE

Calculating the expected energy capture, or production, of wind turbines in an array is the basic function of openWind. To produce an accurate energy capture estimate, the spatial and temporal distribution of the wind resource must be known. This is generally provided in the form of a wind resource grid (WRG), a table of speed and direction distribution parameters at regularly spaced points. Additional information may be provided in tabular (TAB) files representing measurements at one or more meteorological masts.

The characteristics of the wind turbines, including most importantly their power production over a range of incident free-stream wind speeds (power curve), complete the main input elements for the calculation.

1.1 THE ENERGY SUM

Neglecting plant-level losses, the energy produced by the turbines is of the form

$$Energy = 8766 \sum_{i=1}^{nD} \sum_{j=1}^{nU} \sum_{k=1}^{nT} F_{ijk} P_{ijk}$$
(eqn. 1)

where nD is the number of direction steps (default 72), nU is the number of wind speed steps, and nT is the number of turbines. F_{ijk} is the frequency or probability of the wind coming from direction sector i at wind speed bin j at turbine k, and P_{ijk} is the power (in kilowatts) generated by that turbine for the same wind speed and direction.

Multiplying the power by 8766, the average number of hours in one year accounting for leap years, converts power into energy and the sum becomes the number of kilowatt-hours (kWh) generated in the average year.

For a typical energy capture calculation, there are of the order of 72 wind direction steps, between 20 and 140 wind speed steps, and as many turbines as are being modeled in the wind farm, resulting in anywhere from 1000 to 10,000,000 elements in the sum.

The first direction considered in the energy capture is direction 0, which is centered on 0 degrees bearing (due north). The first sector in any WRG or TAB file loaded into openWind is also considered to be centered on zero degrees. Subsequent direction sectors are assumed to progress in a clockwise manner in equal intervals. Speed bins are similarly numbered in equal intervals starting from the lowest speed.

1.2 PROBABILITY

The probability of occurrence of a particular speed or direction is considered over the entire wind speed or wind direction interval. This is in contrast to merely interrogating the distribution at the mid-point, which can lead to incorrect or ill-defined results. For instance, suppose we are modeling a 1 MW turbine with a power curve defined in intervals of 1 m/s and a cut-out wind speed of 25 m/s. What is the power output at 25 m/s? If we merely interrogate the curve at this speed, we obtain either 1000 kW or 0 kW. However, the correct power output over the interval 24.5-25.5 m/s is 500kW, as approximately half the time the wind speed will be in the lower half of the bin, i.e., below the cut-out speed, and the rest of the time it will be in the upper half of the bin.

In openWind, the probability of the wind coming from a certain direction at a certain speed can be calculated in one of two ways, described below.

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1.2.1 WEIBULL FUNCTIONS

Weibull parameters may be obtained from an appropriate wind resource grid (WRG) and can be summed over the wind speed interval. The calculation itself is of the form

$$F = F_{xy\alpha} \left\{ W_{cum}(u + \frac{u_{step}}{2}, A_{xy\alpha}, K_{xy\alpha}) - W_{cum}(u - \frac{u_{step}}{2}, A_{xy\alpha}, K_{xy\alpha}) \right\} \qquad \text{(eqn. 2)}$$

where $F_{xy\alpha}$ is the probability that the wind will come from direction α at the location x,y. The term in brackets is the sum of probability over the interval $\{u-u_{step}/2\}$ to $\{u+u_{step}/2\}$ for the values of A and K at the location x,y. u is the free-stream wind speed.

Example: If direction step α falls 25% within one sector with A=7 m/s and K=2 and 75% in the adjoining sector with A=6 m/s and K=3, then the weighted-average A and K values for direction step α are 6.25 m/s and 2.75 respectively. If the probability F is 0.1 for the first sector and F=0.05 for the second sector, the total probability for this direction step is 0.25*0.1+0.75*0.05=0.0625. For a free-stream wind speed of 8m/s and wind speed interval of 1 m/s, equation 2 gives F=0.0625*{W_cum(8.5,6.25,2.75)-W_cum(7.5,6.25,2.75)}.

W_{cum} is the cumulative Weibull function and is defined as

$$W_{cum}(u, A, K) = 1 - e^{-\left(\frac{u}{A}\right)^K} \qquad \text{(eqn. 3)}$$

A and K are simple averages over the direction interval. They are obtained by examining the sectors of the WRG overlapped by direction step α and creating a sum of all the sector values, each weighted by the relative proportion it represents within the direction step. An example is provided below.

1.2.2 WIND SPEED DISTRIBUTION TABLES

If a wind speed distribution table (TAB file) from a measurement mast is available and associated with the wind resource grid (WRG), the energy capture uses this table to determine the probabilities. The WRG is then used only to determine the average wind speeds at other locations relative to the mast location.

The principal rationale for TAB files is that measured speed and direction distributions are usually more accurate than modeled distributions. In addition, TAB files provide a mechanism for "anchoring," or adjusting, the mean wind resource to measurements at one or more points, which can reduce the overall bias in the energy production estimate.

For this method to work reliably, the TAB file must be associated with a single-point WRG. The single-point WRG represents the WRG interpolated exactly to the mast location. For each direction, the program finds the ratio of the average wind speed at other points in the WRG to that of the single-point WRG;

these ratios are called speed-ups. The speed-ups are then used to adjust the wind resource at the turbine locations in the energy capture loop before being input into the power curve function (see section 3).

1.2.2.1 CALCULATION OF SPEED-UPS

The average wind speed for any pair of Weibull *A* and *K* can be calculated using the gamma function as follows:

$$\bar{u} = A.\Gamma\left(1+\frac{1}{K}\right) \qquad \text{(eqn. 4)}$$

In practice, the natural logarithm of the gamma function is more easily calculated, so the function is implemented as

$$\bar{u} = A.e^{\ln\left(\Gamma\left(1+\frac{1}{K}\right)\right)} \qquad \text{(eqn. 5)}$$

Before equation (1) is executed, speed-ups for each direction step are calculated for each turbine at its current location using the following relationship:

$$SU_{xy\alpha} = \frac{\bar{u}_{xy\alpha}}{\bar{u}_{m\alpha}} \qquad \text{(eqn. 6)}$$

where $SU_{xy\alpha}$ is the speed up at location *x,y* for direction step α, $\bar{u}_{xy\alpha}$ is the mean wind speed from the WRG at location *x,y* and for direction step α, and $\bar{u}_{m\alpha}$ is the mean wind speed at the mast for the same direction step α from the single-point WRG.

1.2.2.2 CALCULATION OF PROBABILITIES FROM TAB FILES

The probability of wind speed *u* and direction α based on the TAB file is the number of occurrences of wind speeds in the range $u-u_{step}/2$ to $u+u_{step}/2$ within the direction bin $\alpha-D_{step}/2$ to $\alpha+D_{step}/2$. This calculation is done in two parts, one nested inside the other. For each TAB sector which the direction step overlaps, the total frequency wind speeds is summed over the range $u-u_{step}/2$ to $u+u_{step}/2$. The frequency calculated for each direction bin is then weighted by the proportion of the target range which is inside that bin and then summed to obtain the total probability of wind within the requested range of wind speed and direction.

1.3   POWER

The power output of a turbine depends on its power curve at a given air density, the incident velocity (accounting for speed-up and wake effects), and the air density at the turbine location (in kg/m^3). The power curve is generally provided by the manufacturer for one or more values of air density.

In the energy sum, the power for a particular speed bin is defined as the average power derived from the power curve over the interval $u-u_{step}/2$ and $u+u_{step}/2$. If no point on the user-defined power curve lies within this interval, then this is the average of the values at the two end points, which are determined by linear interpolation from the bracketing points on the curve. If the power curve contains one or more points between $u-u_{step}$ and $u+u_{step}$ then the averaging is done from $u-u_{step}$ to the first point, then from the first point to the second point, and so on, until $u+u_{step}$ is reached. The weight given to each element in the combined average depends on the width of the corresponding speed interval.

To account for differing air densities, openWind interpolates between power curves defined for two bracketing densities. Where no such bracketing curves are available (e.g., if only one curve is provided or the air density falls outside the range of defined air densities), openWind employs one of two extrapolation techniques depending on the turbine type.[1]

1.3.1   STALL-REGULATED TURBINES

For stall-regulated turbines, the power at a particular incident wind speed is adjusted (raised or lowered) in proportion to the air density.

$$P_{adjusted} = P_0\left(\frac{\rho_{actual}}{\rho_0}\right)$$ (eqn. 7)

where ρ is the air density and P_0 is the power over the wind speed interval $u-u_{step}/2$ to $u+u_{step}/2$ defined for air density ρ_0.

1.3.2   PITCH-REGULATED TURBINES

For pitch-regulated turbines, the power curve is adjusted as follows.

$$P_{adjusted}\left(v_{incident}\right) = P_0\left(v_{adjusted}\right)$$ (eqn. 8)

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[1] These techniques are based on IEC 61400-12.

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Here, $v_{incident}$ is the incident wind speed and $v_{adjusted}$ is given by $v_{adjusted} = v_{incident} \left(\dfrac{\rho_{actual}}{\rho_0} \right)^{\!1/3}$.

In this equation, the power curve is not raised or lowered as with stall-regulated machines; it is shifted to the right or left. This method is based on the assumption that pitch-regulated turbines are able to operate at the same efficiency (extract the same proportion of available wind energy) over a range of air densities and speeds. Since the energy content of the wind is proportional to the air density times the speed cubed, changing the air density by some ratio while keeping the speed constant has the same effect as keeping the air density the same while changing the speed by the cube root of that ratio.

Note that the cut-in and cut-out wind speeds are adjusted by the same factor. However, the wind speed interval in the power curve does not change.

2   WAKE EFFECTS

Wake effects in openWind are represented as modifications to the free-stream wind speed. The modified incident speeds are applied directly in equation (1) to determine the energy capture net of wake losses. openWind presently offers the options of using both the Park and Modified Park wake models.

2.1   PARK MODEL

Park was developed originally by N.O. Jensen (1984) and Katic et al. (1986) and has been implemented in the WAsP software package of Risoe National Laboratory. It assumes an initial velocity deficit immediately behind the turbine rotor, which is calculated from the turbine's thrust coefficient (C_t), and an empirically determined wake-decay constant, which sets the linear rate of expansion of the wake with distance downstream. Park assumes that the wind flow, including the entrained wake, follows the terrain. The effects of multiple wakes are taken into account by superimposing, or overlapping, the wake cross-sections of the upstream turbines. These basic elements are illustrated below.

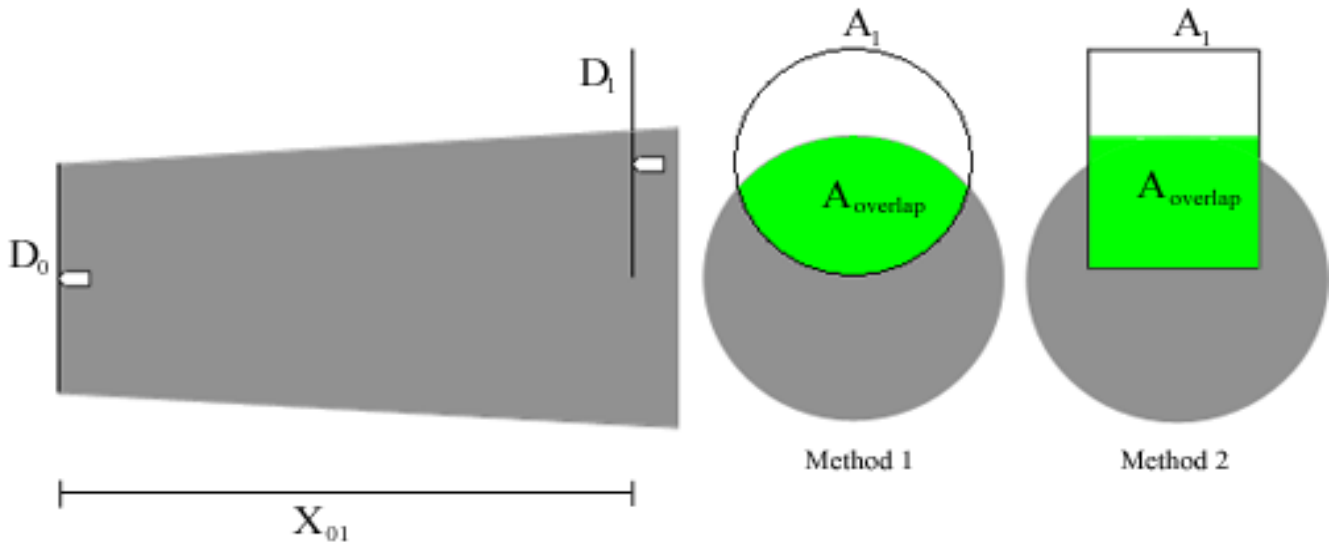OPENWIND THEORETICAL BASIS AND VALIDATION

Figure 2.1 Plan view of turbine 1 partially in the wake of turbine 0.

The combined wind speed deficit at the down-wind turbine is calculated as follows (following the notation of Katic et al. (1986)).

$$\delta V_{01} = U_0 \cdot \left(1 - \sqrt{1 - C_t}\right) \left(\frac{D_0}{D_0 + 2kX_{01}}\right)^2 \cdot \frac{A_{overlap}}{A_1} \qquad \text{(eqn. 9)}$$

Here, U_0 is the incident wind speed at the up-wind turbine ("0") with rotor diameter D_0, C_t is the thrust coefficient, X_{01} the downwind horizontal distance between the wind turbines and k is the wake decay constant.

The incident wind speed at a downwind turbine is the free-stream wind speed minus the wake deficit calculated by equation 9. Where there are several upstream wind turbines with overlapping wakes, the incident speed is assumed to equal the free-stream speed minus the largest single wake deficit.

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2.2   MODIFIED PARK MODEL

Modified Park incorporates changes to the Park model developed by Garrad Hassan and Partners, Ltd. The area of overlap is not calculated as the area of intersection of the wake and the incident rotor disk, as in Park (method 1). Instead, the fraction of horizontal overlap is calculated at right-angles to the wind direction but parallel to the ground, and the velocity deficit is multiplied by this fraction (method 2). Furthermore, in Modified Park the wake generated by a wake-affected turbine is the same as if the turbine were in the free-stream except that its thrust coefficient, C_t, is calculated using the incident wind speed for that turbine. In general, these modifications produce a smaller wake loss than that estimated by Park.

2.3   EDDY-VISCOSITY WAKE MODEL

The eddy-viscosity wake model in openWind implements the model described in Ainslie (1988). The derivation of the numerical solution scheme is given in Appendix A. The model represents the wake-induced wind-speed deficit as a Gaussian curve initially two rotor diameters (2D) downstream with its peak directly in line with the turbine axis. This initial velocity deficit is modified by mixing with the free-stream wind around the wake. The rate of mixing is determined by the ambient turbulence intensity – the

> **Example:** Two turbines with rotor diameters of 80 m are facing directly into the wind, which is coming from the west. Turbine B is 200 meters to the east of Turbine A and 50 meters to the north, putting it partially in Turbine A's wake. The wake decay constant is 0.1, so the diameter of Turbine A's wake at Turbine B calculated by Modified Park is (from equation 9) 80m + 2 x 0.1 x 200 = 120m. Its radius is therefore 60 m. Considering Turbine B's radius of 40m and its 50 m offset from the wake center line, the overlap fraction according to Modified Park is (60+40-50)/80 = 0.625.

greater the ambient turbulence intensity, the greater the rate of mixing and the faster the wake deficit tends to zero. The initial centerline velocity deficit (relative to the speed at the rotor inlet) is given by

$$D_{mi} = C_t - 0.05 - (16C_t - 0.5)\frac{I_0}{1000}$$

(eqn.10)

where D_{mi} is the initial centerline velocity; C_t is the thrust coefficient (a function of the incident wind speed); and I_0 is the ambient turbulence intensity in percent.

This deficit is then scaled by the ratio of the free-stream velocity, U_0, to the incident velocity, U_i, in order to transform the wake wind velocity deficit into a fraction of the free-stream velocity.

The wake width increases with downstream distance as the radial wake deficit profile gradually flattens. The wake width at any point is given by

$$b = \sqrt{\frac{3.56 C_t}{8 D_m (1 - 0.5 D_m)}}$$

(eqn. 11)

where b is a measure of the wake width and D_m is the centerline velocity deficit at any point.

The Eddy Viscosity model predicts the velocity deficit along the centerline, which is then used to describe a Gaussian-shaped velocity profile with the width given in equation 11 for any point downstream. The Gaussian profile is of the form

$$D = U_i \left(1 - D_m e^{-3.56 \frac{r^2}{b^2}} \right)$$

(eqn. 12)

where D is the deficit at any point, U_i is the incident velocity entering the rotor, r is the radial distance from the centerline, b is the wake width, and D_m is the centerline velocity deficit.

The incident turbulence intensity at each turbine in the eddy viscosity wake calculation is taken to be the ambient turbulence intensity and this value is then used for the entire downstream wake calculation. The turbulence intensity influences the eddy viscosity calculation through the eddy viscosity term in the following manner:

$$\varepsilon = F K_1 b (U_0 - U_c) + K_m$$

(eqn. 13)

and

$$K_m = F \kappa^2 \frac{I_0}{100}$$

(eqn. 14)

where ε is the eddy viscosity; K_1 is a dimensionless constant set equal to 0.015; b is the wake width; U_0 is the free-stream velocity; U_c is the wake centerline velocity; κ is the Von Karman constant; I_0 is the ambient turbulence intensity in percent; and F is an optional filter function such that

$$F = 1.0 \text{ for } x > 5.5$$
$$F = 0.65 + \left(\frac{X - 4.5}{23.32} \right)^{1/3} \text{ for } x < 5.5$$

(eqn. 15)

where x is the downstream distance from the rotor in rotor diameters. When the filter function is disabled, F=1.0.

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At each downstream step, the above functions are integrated using the Crank-Nicolson differencing scheme to predict the deficit at the next step. For details, see Appendix A.

2.4 TURBULENCE INTENSITY DUE TO WAKES

Aside from a deficit in wind speed, turbine wakes are characterized by an increase in turbulence. This can reduce the performance of turbines downstream as the associated fluctuations in speed and direction may be too rapid for the turbines to respond (in either yaw or pitch) and maintain optimal output. It can also cause greater wear and tear on turbine components.

Neither Park nor Modified Park calculates the turbulence intensity. Their equations depend on empirical relationships that do not include turbulence. OpenWind, however, contains equations for estimating the combined ambient and wake-induced turbulence and its impact on turbine output.

The turbulence intensity (TI) is defined as the standard deviation of fluctuations in speed divided by the mean speed. The ambient TI is typically measured by anemometers or other instruments and entered by the user as a table of values as a function of speed. It is assumed to vary across the site by the inverse of the speed-up. This reflects the assumption that topography affects only the mean speed, not fluctuations in speed caused by turbulent eddies.

Wake-induced turbulence is caused by vortices created at the blade tips and close to the hub as well as flow distortions through and around the rotor disk. To estimate its contribution to the total turbulence, openWind implements the following experimentally-derived equation (Hassan 1992; Vermeulen 1980):

$$I_+ = 5.7.C_t^{0.7}.I_0^{0.68}.(x/x_n)^{-0.96}$$
(eqn. 16)

I_+ is the wake-induced turbulence intensity, I_0 is the ambient (or incident) turbulence intensity, x is the downstream distance from the turbine creating the wake, and x_n is the length of the near wake (explained below). The term x/x_n is limited such that its value is never allowed to fall below 1.5.

Since wake-induced and ambient speed fluctuations are assumed to be normally distributed and uncorrelated in time, the total turbulence intensity can be calculated as

$$I_{Total} = \frac{U_0}{U_i}\sqrt{P_c I_+^2 + I_0^2}$$
(eqn. 17)

where U_0 is the free-stream wind speed, U_i the incident wind speed and P_c is the fraction of the rotor disk judged to be in the wake. The distribution of wake-induced turbulence intensity is modeled as constant across the wake and the wake width is considered to be twice the eddy viscosity width, b, or the wake width when using Park or Modified Park.

The near wake is the region just behind and a little outside the edge of the rotor disk in which the wake retains an asymmetric internal structure, including helical vortices. In the far wake, the wake is assumed

to be homogeneous with axial symmetry (ignoring the influence of the ground and the presence of wind shear above the top of the rotor disk). From Vermeulen (1981:2), the near-wake length is defined as

$$x_n = \frac{nr_0}{\left(\dfrac{dr}{dx}\right)} \qquad \text{(eqn. 18)}$$

where

$$r_0 = R\sqrt{\frac{m+1}{2}} \qquad \text{(eqn. 19)}$$

$$m = \frac{1}{\sqrt{1-C_t}} \qquad \text{(eqn. 20)}$$

$$n = \frac{\sqrt{0.214 + 0.144m}.\left(1 - \sqrt{0.134 + 0.124m}\right)}{\left(1 - \sqrt{0.214 + 0.144m}\right)\sqrt{0.134 + 0.124m}} \qquad \text{(eqn. 21)}$$

and dr/dx is the wake growth rate:

$$\frac{dr}{dx} = \sqrt{\left(\frac{dr}{dx}\right)_\alpha^2 + \left(\frac{dr}{dx}\right)_m^2 + \left(\frac{dr}{dx}\right)_\lambda^2} \qquad \text{(eqn. 22)}$$

Here,

$$\left.\begin{cases}\left(\dfrac{dr}{dx}\right)_\alpha = 0.05.I_0\end{cases}\right\}I_0 < 2 \\[2em] \left.\begin{cases}\left(\dfrac{dr}{dx}\right)_\alpha = 0.025.I_0 + 0.05\end{cases}\right\}I_0 \geq 2 \qquad \text{(eqn. 23)}$$

is the growth rate due to ambient turbulence,

$$\left(\frac{dr}{dx}\right)_m = \frac{(1-m)\sqrt{1.49+m}}{(1+m)9.76} \qquad \text{(eqn.24)}$$

is the growth rate due to shear-generated turbulence, and

$$\left(\frac{dr}{dx}\right)_\lambda = 0.012B\lambda \qquad \text{(eqn.25)}$$

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is the contribution due to mechanical turbulence, where B is the number of blades and λ is the tip speed ratio.

The tip speed ratio is defined as

$$\lambda = \frac{RPM(u, \rho).\pi.D}{60v}$$ (eqn. 26)

In this equation, *RPM* is the rotation rate of the turbine, which is a function of the wind speed and the air density (openWind includes the option to define *RPM* as a function of air density in the same way as power), *D* is the rotor diameter, and *v* is the incident wind speed.

Note that the wake growth rate for turbulence intensity may be different from the wake growth rate factor *K* in the Park and Modified Park models.

2.5   TURBULENCE INTENSITY LOSS

openWind has the ability to impose a loss related to turbulence intensity. It is described as a percentage reduction in output for every percent increase in the total turbulence intensity (ambient plus wake effects) at each turbine in excess of that specified for the turbine power curve. By convention, power curves are defined for a given range of TI values at a mean speed of 15m/s. The same convention is followed by openWind for calculating the TI loss.

3   NON-WAKE ENERGY LOSSES

Wind power plants can experience many other losses besides those caused by wake effects. Examples include electrical losses, turbine downtime, and blade soiling and icing. In openWind, such losses are applied cumulatively in the form:

$$E_{net} = E_{gross}.(1 - L_1).(1 - L_2).(1 - L_3)......(1 - L_n)$$ (eqn. 27)

Such non-wake losses are specified by the user and require no further explanation.

4   AIR DENSITY

Air density impacts turbine performance and is an input to the power calculation (see section 1.3). According to the ideal gas law, the air density is related inversely to temperature and directly to air pressure. Since on-site air pressure measurements are uncommon, however, the usual practice is to estimate the air density from the temperature and elevation.

For each site, the user can specify the elevation as well as either the air temperature or the air density. The air density variation can be calculated using the air density lapse rate or the air temperature lapse rate.

4.1 AIR TEMPERATURE LAPSE RATE

Over most practical ranges of elevation, the air temperature varies almost perfectly linearly with elevation above sea level:

$$T_{turbine} = T_{site} + T_{lapse}\left(Z_{turbine} - Z_{site}\right)$$ (eqn. 28)

Here, T is the temperature and Z is elevation above sea level.

4.2 AIR DENSITY LAPSE RATE

Alternatively, it may be assumed that the air density varies linearly with elevation. While this is a good assumption in many circumstances, the air temperature lapse rate is more uniform.

$$\rho_{turbine} = \rho_{site} + \rho_{lapse}\left(Z_{turbine} - Z_{site}\right)$$ (eqn. 29)

4.3 CONVERSION BETWEEN AIR TEMPERATURE AND AIR DENSITY

The conversion between air temperature and air density rests on assumptions regarding the temperature and air density profile throughout the atmosphere. Where it is necessary to make this conversion, openWind implements the following equation.

$$\rho = \frac{101325e^{\left(\frac{-9.8Z}{287.05(273.15+T)}\right)}}{287.05(T + 273.15)}$$ (eqn. 30)

This assumes the air density decreases exponentially with elevation, which implies an isothermal temperature profile. Although this assumption does not hold true in general, the equation is nevertheless quite accurate for most elevations of interest for wind energy.

5 VALIDATION

The equations and assumptions described in the foregoing sections are well known in the wind industry. To verify that they have been implemented correctly in openWind, AWS Truewind has carried out a comprehensive model-to-model validation exercise involving 20 real-world wind projects, using as a control the WindFarmer program of Garrad Hassan and Partners, Ltd. The following presents the results for the gross energy capture and wake losses.

5.1   ENERGY CAPTURE AND WAKE LOSSES

5.1.1   ASSUMPTIONS

To ensure comparable results, the energy capture and wake loss tests employed the same WRG and TAB files in each program. The same turbine power and thrust coefficient curves were also used, along with an identical site reference air density (defined for a reference height considered representative of the elevations of the turbines). The same turbine layouts were tested in each program.

Three sets of tests were carried out

1. *Modified Park wake model without air density adjustment*. WindFarmer uses a proprietary method to adjust power curves to take account of variations in air density. For these tests, the air density adjustment (ADA) was switched off in both WindFarmer and openWind by setting the air density lapse rate to zero. The Modified Park model was used by both programs.

2. *Modified Park wake model with air density adjustment*. These tests were identical to the first except that the air density adjustment was turned on in both programs, with a lapse rate of -0.113kg/m^3/km.

3. *Eddy Viscosity wake model with air density adjustment*. These tests were the same as the previous except that the Eddy Viscosity model was used to estimate array losses.

For all tests, in WindFarmer, in the site properties on the "energy" tab, the "direction shift of sector probabilities" was switched off (as this option does not exist in openWind) and the "topography correction to wake model" option was switched on. The number of direction steps and wind speed steps were kept at their default values of 72 and 70, respectively, in both openWind and WindFarmer.

With the Modified Park model, cases were run both with and without association of the WRG to a TAB file, providing a test of the different methods of calculating sector-wise probabilities and mean speeds.

With the Eddy Viscosity model, due to the need to include ambient turbulence intensity, tests were only run with associated TAB files. Each of the twenty sets of site data used to validate the Eddy Viscosity model included a table of ambient turbulence intensity by direction and wind-speed.

5.1.2   RESULTS

Tables 5.1 and 5.2 present the results of the initial energy capture and wake loss tests using Modified Park with the air density lapse rate disabled. When using TAB files, the mean discrepancy (openWind minus WindFarmer) in gross energy was -0.01%, with a standard deviation of 0.07%. The largest positive discrepancy (site 11) was 0.10%, and the largest negative discrepancy (site 17) was -0.12%. For the net energy including wake losses, the statistics were comparable: a mean discrepancy of -0.02% with a

standard deviation of 0.07%. The wake losses themselves are virtually identical (mean discrepancy 0.0% of energy, standard deviation 0.02% of energy).

For the runs without TAB files, the results were very similar. The mean discrepancies in gross and net energy capture were -0.03% and -0.03%, indicating a slight tendency for openWind to predict less energy production than WindFarmer under the same conditions. The standard deviations of the discrepancies were 0.07% in both cases, confirming that the mean difference, though very small, is statistically significant. Again, the wake losses themselves are virtually identical (mean discrepancy 0.0% of energy, standard deviation 0.02% of energy).

The differences in energy capture between the two models are so small that we conclude that the calculations are functionally identical for both methods of energy capture as well as the Modified Park wake model.

When we repeated these tests with the air density adjustment switched on, the deviations were slightly larger. When using TAB files, the mean discrepancy in gross energy was -0.04%, with a standard deviation of 0.12%. For the net energy including wake losses, the statistics were comparable: a mean discrepancy of -0.04% with a standard deviation of 0.12%. (The wake loss results were the same as above as this part of the calculation is not affected by the air density adjustment.)

Without TAB files, the mean discrepancy in gross energy was -0.06%, with a standard deviation of 0.12%. For the net energy including wake losses, the statistics were comparable: a mean discrepancy of -0.06% with a standard deviation of 0.12%.

We conclude that the air density adjustment carried out by openWind is, on the whole, conservative by comparison with that in WindFarmer.

The same tests were carried out using the Eddy Viscosity wake model with the air density adjustment turned on. As can be seen from Table 5.3, the openWind Eddy Viscosity wake model consistently estimates a greater wake loss compared to the WindFarmer version of the same model. Chart 5.1 shows that there is a strong relationship between the wake losses predicted by the two models, with a mean discrepancy of about 17%. The explanation for this discrepancy is unknown. As noted, openWind implements the original Ainslie model; it is possible the WindFarmer version has been modified in some way. AWS Truewind is in the process of comparing the wake estimates of both models against historical plant output data.

The comparison of openWind against WindFarmer also included turbulence intensity. The results are given in Table 5.4. openWind's estimate is typically about 0.2-0.3 percentage points greater than WindFarmer's, a difference that appears to be attributable to the calculation of the ambient intensity.

Table 5.1 Comparison of energy capture and Modified Park wake losses with TAB files without the air density adjustment.

	WindFarmer			openWind		
Site	Gross (GWh)	Net (GWh)	Wake Loss (%)	Gross (GWh)	Net (GWh)	Wake Loss (%)
1	179.5	170.4	5.1	179.5	170.4	5.1

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2	473.5	459.8	2.9	473.6	459.8	2.9
3	388.3	363.8	6.3	388.7	364.0	6.3
4	200.4	193.5	3.4	200.3	193.5	3.4
5	409.8	386.1	5.8	410.0	386.3	5.8
6	973.1	918.2	5.6	972.5	917.5	5.7
7	170.9	160.1	6.3	170.8	160.0	6.3
8	149.7	143.6	4.1	149.7	143.6	4.1
9	611.6	581.4	4.9	611.1	580.8	5.0
10	357.8	337.4	5.7	357.6	337.2	5.7
11	962.7	916.2	4.8	963.7	917.0	4.8
12	87.3	86.1	1.4	87.2	86.0	1.4
13	578.6	553.5	4.3	578.2	553.2	4.3
14	197.9	184.6	6.8	197.9	184.5	6.8
15	383.1	366.9	4.2	383.5	367.2	4.3
16	613.0	570.1	7.0	613.0	570.2	7.0
17	194.4	189.8	2.4	194.2	189.5	2.4
18	2557.4	2451.8	4.1	2555.4	2449.3	4.2
19	2699.0	2578.3	4.5	2698.9	2579.3	4.4
20	227.1	215.8	5.0	227.3	216.0	5.0

Table 5.2 Comparison of energy capture and Modified Park wake losses without TAB files without the air density adjustment.

Site	WindFarmer			openWind		
	Gross (GWh)	Net (GWh)	Wake Loss (%)	Gross (GWh)	Net (GWh)	Wake Loss (%)
1	181.8	172.9	4.9	181.8	172.9	4.9
2	461.5	447.8	3.0	461.5	447.8	3.0
3	373.1	349.8	6.2	373.3	350.0	6.3
4	193.6	187.0	3.4	193.5	186.9	3.4
5	389.3	364.4	6.4	389.4	364.5	6.4
6	952.6	896.7	5.9	952.0	896.1	5.9
7	167.3	157.0	6.1	167.2	156.9	6.2
8	145.5	139.5	4.1	145.4	139.4	4.1
9	585.0	555.7	5.0	584.3	554.9	5.0
10	353.5	333.9	5.6	353.4	333.7	5.6
11	947.3	898.7	5.1	948.1	899.2	5.2
12	70.7	69.6	1.5	70.6	69.5	1.5
13	535.4	510.9	4.6	534.8	510.5	4.6
14	199.3	186.8	6.3	199.2	186.8	6.3
15	371.9	355.7	4.3	372.1	355.9	4.4
16	592.5	551.3	7.0	592.5	551.5	6.9
17	187.9	183.5	2.3	187.7	183.2	2.4
18	2457.9	2350.4	4.4	2454.9	2346.9	4.4
19	2394.8	2277.8	4.9	2394.8	2279.0	4.8
20	218.8	207.9	5.0	219.0	208.0	5.0

Table 5.3 Comparison of energy capture and Modified Park wake losses with TAB files with air density adjustment.

Site	WindFarmer			openWind		
	Gross (GWh)	Net (GWh)	Wake Loss (%)	Gross (GWh)	Net (GWh)	Wake Loss (%)
1	179.5	170.4	5.1	179.5	170.4	5.1
2	473.5	459.8	2.9	473.6	459.8	2.9
3	388.3	363.7	6.3	388.6	364.0	6.3
4	200.3	193.5	3.4	200.3	193.4	3.4
5	409.8	386.0	5.8	409.9	386.2	5.8
6	973.5	918.6	5.6	972.7	917.7	5.7
7	170.8	160.1	6.3	170.7	159.9	6.3
8	149.7	143.6	4.1	149.7	143.6	4.1
9	611.6	581.4	4.9	611.1	580.8	5.0
10	337.6	318.4	5.7	336.1	317.0	5.7
11	962.7	916.2	4.8	963.7	917.0	4.8
12	87.3	86.1	1.4	87.2	86.0	1.4
13	578.5	553.4	4.3	578.1	553.1	4.3
14	197.9	184.5	6.8	197.9	184.5	6.8
15	383.1	366.9	4.2	383.5	367.2	4.3
16	613.0	570.1	7.0	613.0	570.2	7.0
17	194.4	189.8	2.4	194.2	189.6	2.4
18	2556.6	2451.0	4.1	2554.5	2448.5	4.2
19	2705.8	2585.0	4.5	2705.1	2585.4	4.4
20	226.6	215.3	5.0	226.8	215.5	5.0

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Table 5.4 Comparison of energy capture and Modified Park wake losses without TAB files with air density adjustment.

Site	WindFarmer			openWind		
	Gross (GWh)	Net (GWh)	Wake Loss (%)	Gross (GWh)	Net (GWh)	Wake Loss (%)
1	181.8	172.9	4.9	181.8	172.9	4.9
2	461.5	447.8	3.0	461.5	447.8	3.0
3	373.0	349.8	6.2	373.3	349.9	6.2
4	193.6	187.0	3.4	193.5	186.9	3.4
5	389.2	364.3	6.4	389.3	364.4	6.4
6	952.9	897.0	5.9	952.1	896.2	5.9
7	167.2	157.0	6.1	167.1	156.8	6.2
8	145.5	139.5	4.1	145.5	139.4	4.1
9	585.0	555.7	5.0	584.3	554.9	5.0
10	334.1	315.6	5.5	332.6	314.2	5.5
11	947.3	898.7	5.1	948.1	899.2	5.2
12	70.7	69.6	1.5	70.5	69.4	1.5
13	535.3	510.8	4.6	534.7	510.4	4.6
14	199.3	186.8	6.3	199.2	186.8	6.3
15	371.9	355.7	4.3	372.1	355.9	4.4
16	592.5	551.3	7.0	592.5	551.4	6.9
17	187.9	183.5	2.3	187.7	183.3	2.4
18	2457.1	2349.9	4.4	2454.1	2346.1	4.4
19	2400.3	2283.0	4.9	2399.2	2283.1	4.8
20	218.3	207.4	5.0	218.5	207.5	5.0

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Table 5.5 Comparison of energy capture and wake losses using the Eddy Viscosity wake model.

Site	WindFarmer			openWind		
	Gross (GWh)	Net (GWh)	Wake Loss (%)	Gross (GWh)	Net (GWh)	Wake Loss (%)
1	179.5	170.9	4.8	179.5	169.3	5.7
2	473.5	460.3	2.8	473.6	458.2	3.2
3	388.3	366.9	5.5	388.6	363.0	6.6
4	200.3	193.3	3.5	200.3	192.3	4.0
5	409.8	387.5	5.4	409.9	383.5	6.4
6	973.5	926.4	4.8	972.7	917.1	5.7
7	170.8	160.7	6.0	170.7	158.4	7.2
8	149.7	143.9	3.9	149.7	143.0	4.4
9	611.6	579.1	5.3	611.1	573.9	6.1
10	337.6	318.1	5.8	336.1	312.9	6.9
11	962.7	920.5	4.4	963.7	915.1	5.0
12	87.3	86.0	1.5	87.2	85.9	1.4
13	578.5	548.8	5.1	578.1	546.7	5.4
14	197.9	186.9	5.6	197.9	183.8	7.1
15	383.1	366.5	4.4	383.5	364.7	4.9
16	613.0	571.9	6.7	613.0	565.3	7.8
17	194.4	189.9	2.3	194.2	188.4	3.0
18	2556.6	2441.8	4.5	2554.5	2429.1	4.9
19	2705.8	2583.6	4.5	2705.1	2570.1	5.0
20	226.6	215.5	4.9	226.8	213.6	5.8

openWind Eddy Viscosity Wake Losses Plotted Against WindFarmer Eddy Viscosity Wake Losses

y = 1.1685x
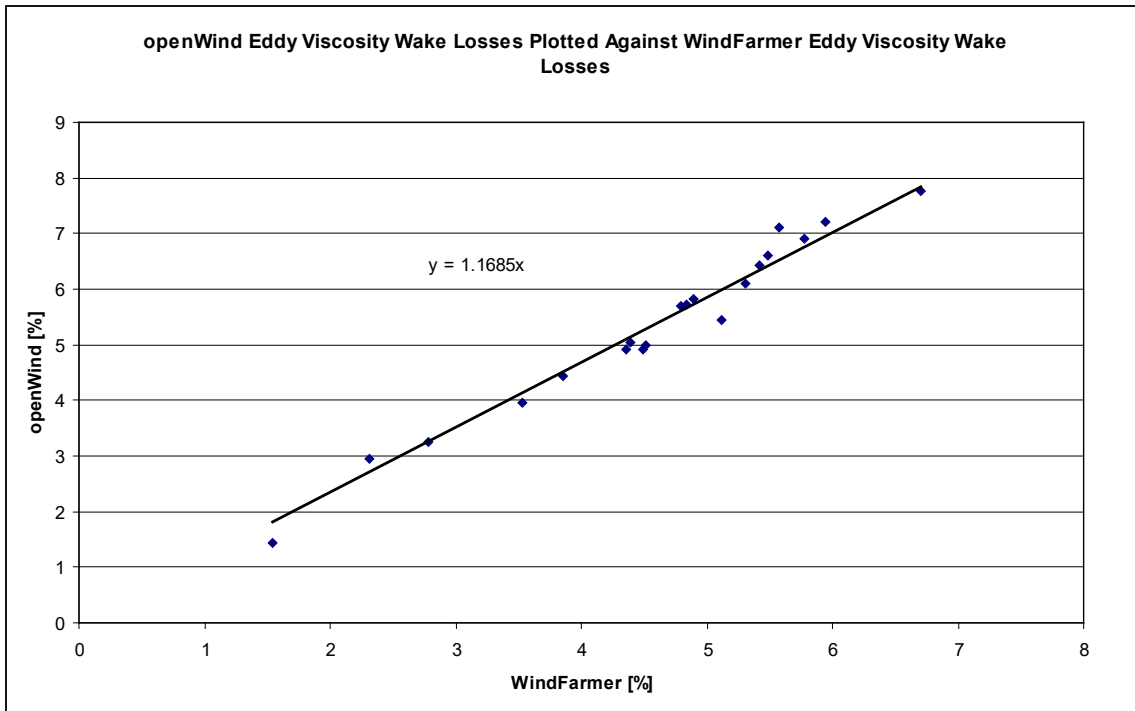
Chart 5.1: openWind Eddy Viscosity wake losses plotted against WindFarmer Eddy Viscosity wake losses.

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Table 5.6 Comparison of turbulence intensity and wake losses using the Eddy Viscosity wake model.

	WindFarmer			openWind		
Site	Wake Loss (%)	Ambient TI (%)	Total TI (%)	Wake Loss (%)	Ambient TI (%)	Total TI (%)
1	4.8	12.7	13.7	5.7	12.9	14.0
2	2.8	13.8	14.5	3.2	13.9	14.7
3	5.5	15.4	16.9	6.6	15.5	17.2
4	3.5	10.7	11.3	4.0	10.8	11.4
5	5.4	12.9	14.8	6.4	13.0	14.8
6	4.8	17.6	19.1	5.7	17.8	19.1
7	6.0	12.2	13.8	7.2	12.3	14.0
8	3.9	11.9	12.7	4.4	12.1	12.8
9	5.3	9.9	10.7	6.1	10.0	10.9
10	5.8	13.0	15.9	6.9	13.1	15.6
11	4.4	12.9	14.0	5.0	13.0	14.2
12	1.5	20.7	21.7	1.4	19.1	20.0
13	5.1	16.3	17.4	5.4	16.8	17.5
14	5.6	13.2	15.4	7.1	13.4	15.3
15	4.4	9.6	10.7	4.9	9.7	10.7
16	6.7	9.1	10.5	7.8	9.2	10.7
17	2.3	11.5	12.0	3.0	11.6	12.3
18	4.5	10.6	11.1	4.9	10.7	11.3
19	4.5	11.3	12.8	5.0	11.4	12.5
20	4.9	10.6	12.0	5.8	10.7	12.0

6   CONCLUSIONS

OpenWind is a new wind farm design and optimization program developed by AWS Truewind, LLC. The software, which is built in an open-source environment and uses GIS data types and a GIS-style interface, represents a substantial departure from programs that have preceded it. However, the methods employed to calculate the energy capture and wake losses rest on the same theoretical foundations, as described in this document.

To establish consistency with other leading programs, AWS Truewind has carried out a comprehensive model-to-model validation exercise involving 20 wind projects and the WindFarmer program of Garrad Hassan and Partners, Ltd. It is found that for the Modified Park wake model, using the same wind resource and turbine model and with the same program settings, the two produce functionally identical results, with a mean variance of 0.07%.

A further set of tests were carried out using the Eddy Viscosity wake model and another 20 wind projects. It was found that the Eddy Viscosity model in openWind gives approximately 17% higher wakes losses than that in WindFarmer.

6 GLOSSARY OF TERMS

P – power in kW

F – frequency, also referred to as probability throughout this document as all frequencies are converted to values between 0 and 1.

E – energy in kWh

u – the "free-stream" wind speed (the wind speed without wake effects)

v – the "incident" wind speed (the wind speed considering the effects of upstream turbines)

D – the rotor diameter (usually twice the blade length or a little more)

L – energy loss

X – downstream distance

Y – across stream distance (used to calculate whether one turbine is in the wake of another and if so by how much)

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APPENDIX A – NUMERICAL SOLUTION OF EDDY VISCOSITY MODEL

As given in Ainslie (1988),

$$U\frac{\delta U}{\delta x} + V\frac{\delta U}{\delta r} = \frac{\varepsilon}{r}\frac{\delta(r\,\delta U/\delta r)}{\delta r}$$

where

U is the velocity in the downstream direction

V is the velocity in the radial direction

x is a measure of downstream distance (subscript j)

r is a measure of radial distance (subscript i)

ε is the eddy viscosity which is known for each x=j

Using the product rule this becomes

$$U\frac{\delta U}{\delta x} + V\frac{\delta U}{\delta r} = \frac{\varepsilon}{r}\left(\frac{\delta U}{\delta r} + r\frac{\delta^2 U}{\delta r^2}\right)$$

To solve this by finite-differencing, we employ the Crank-Nicolson centre-differencing scheme:

$$U_{i,j}\left(\frac{U_{i,j+1}-U_{i,j}}{k}\right)+\frac{V_{i,j}}{2}\left(\frac{U_{i+1,j+1}-U_{i-1,j+1}}{2h}+\frac{U_{i+1,j}-U_{i-1,j}}{2h}\right)$$

$$=\frac{\varepsilon}{2r}\left(\frac{U_{i+1,j+1}-U_{i-1,j+1}}{2h}+\frac{U_{i+1,j}-U_{i-1,j}}{2h}\right)+\frac{\varepsilon}{2}\left(\frac{U_{i+1,j+1}-2U_{i,j+1}+U_{i-1,j+1}}{h^2}+\frac{U_{i+1,j}-2U_{i,j}+U_{i-1,j}}{h^2}\right)$$

(eqn. A1)

where h = δr and k = δx.

Time is replaced by downstream distance because we are modeling a steady state wind field.

To solve this equation we start with a known initial state (wake deficit as a function of radial distance from the center line) at a given distance downstream of the rotor, and then step farther downstream (or forward in time). At a given radial distance i, and for a known $U_{i+1,j}$, $U_{i,j}$, and $U_{i-1,j}$, we have the following unknowns: $U_{i+1,j+1}$, $U_{i,j+1}$, $U_{i-1,j+1}$ and $V_{i,j}$.

We multiply by $4rkh^2$, then put all the U_{j+1} terms on the left and all the U_j terms on the right.

$$4rh^2 U_{i,j}U_{i,j+1}+rhkV_{i,j}(U_{i+1,j+1}-U_{i-1,j+1})-2rk\varepsilon(U_{i+1,j+1}-2U_{i,j+1}+U_{i-1,j+1})-hk\varepsilon\left(U_{i+1,j+1}-U_{i-1,j+1}\right)$$

$$=hk\varepsilon\left(U_{i+1,j}-U_{i-1,j}\right)+2rk\varepsilon\left(U_{i+1,j}-2U_{i,j}+U_{i-1,j}\right)-rhkV_{i,j}(U_{i+1,j}-U_{i-1,j})+4rh^2 U_{i,j}^2$$

The left-hand side then can be rearranged

$$k\left(h\varepsilon-rhV_{i,j}-2r\varepsilon\right)U_{i-1,j+1}+4r\left(h^2 U_{i,j}+k\varepsilon\right)U_{i,j+1}+k\left(rhV_{i,j}-2r\varepsilon-h\varepsilon\right)U_{i+1,j+1} \quad \text{(eqn. A2)}$$

This is a tri-diagonal matrix, which can be solved by the Thomas algorithm. In the standard notation

$$a_i U_{i-1,j+1}+b_i U_{i,j+1}+c_i U_{i+1,j+1}=r_i \quad\quad\quad\quad\text{(eqn. A3)}$$

where a_i = $k\left(h\varepsilon-rhV_{i,j}-2r\varepsilon\right)$, b_i = $4r\left(h^2 U_{i,j}+k\varepsilon\right)$, c_i = $k\left(rhV_{i,j}-2r\varepsilon-h\varepsilon\right)$, and

$$r_i=hk\varepsilon\left(U_{i+1,j}-U_{i-1,j}\right)+2rk\varepsilon\left(U_{i+1,j}-2U_{i,j}+U_{i-1,j}\right)-rhkV_{i,j}(U_{i+1,j}-U_{i-1,j})+4rh^2 U_{i,j}^2$$

(eqn. A4)

Equation A2 can be re-written in matrix notation as

$$A\underset{\sim}{U}=\underset{\sim}{r} \quad\quad\quad\quad\text{(eqn. A5)}$$

where the vectors are

$$
U \equiv \begin{vmatrix} U_0 \\ U_1 \\ \vdots \\ U_{I-2} \\ U_{I-1} \end{vmatrix} \quad r \equiv \begin{vmatrix} r_0^* \\ r_1 \\ \vdots \\ r_{I-2} \\ r_{I-1}^* \end{vmatrix}
$$

and the tri-diagonal matrix is

$$
A = \begin{vmatrix}
b_0 & c_0 & & & & 0 \\
a_1 & b_1 & c_1 & & & \\
& \cdot & \cdot & \cdot & & \\
& & \cdot & \cdot & \cdot & \\
& & & a_{I-2} & b_{I-2} & c_{I-2} \\
0 & & & & a_{I-1} & b_{I-1}
\end{vmatrix}
$$

The Thomas algorithm fills in $\underset{\sim}{U}$ based on A and $\underset{\sim}{r}$. We deal with the boundary conditions below. By the end we will have all the values of U at x = j+1 based on the values of $r_{i=0\ to\ I-1}$ which is based on the values of U and V at x = j.

Due to radial symmetry, at the wake centerline where r = i = 0, dU/dr = 0 and equation 1 simplifies to become

$$
U_{i,j}\left(\frac{U_{i,j+1} - U_{i,j}}{k}\right) = \frac{\varepsilon}{2}\left(\frac{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}}{h^2} + \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}\right)
$$

We multiply by 2kh² to get

$$
2h^2 U_{i,j}(U_{i,j+1} - U_{i,j}) = k\varepsilon(U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1} + U_{i+1,j} - 2U_{i,j} + U_{i-1,j})
$$

Then

$$
-k\varepsilon U_{i-1,j+1} + 2(h^2 U_{i,j} + k\varepsilon)U_{i,j+1} - k\varepsilon U_{i+1,j+1} = k\varepsilon(U_{i+1,j} - 2U_{i,j} + U_{i-1,j}) + 2h^2 U_{i,j}^2 \qquad \text{(eqn. A6)}
$$

The only unknown on the right hand side is the radial velocity $V_{i,j}$. This is found from the continuity equation,

$$
\frac{\delta u}{\delta x} + \frac{1}{r}\frac{\delta(rv)}{\delta r} = 0
$$

Using the product rule this becomes

$$\frac{\delta u}{\delta x} = -\frac{1}{r}\left(r\frac{\delta v}{\delta r} + v\frac{\delta r}{\delta r}\right)$$

This rearranges to give

$$v = -r\left(\frac{\delta v}{\delta r} + \frac{\delta u}{\delta x}\right)$$

which, choosing to take a backward difference, differences as

$$v_{i,j} = -r\left(\frac{v_{i,j} - v_{i-1,j}}{h} + \frac{u_{i,j} - u_{i,j-1}}{k}\right)$$

where h = dr and k = dx. After some manipulation this rearranges to give

$$v_{i,j} = \frac{r}{r+h}\left(v_{i-1,j} - \frac{h}{k}(u_{i,j} - u_{i,j-1})\right) \qquad \text{(eqn. A7)}$$

The final step is to incorporate the boundary conditions. At the edge of the wake, $U=U_o$, the free-stream velocity. This boundary condition is incorporated by moving the c term over to the right-hand side for the last row of the tri-diagonal matrix where it becomes subtracted from r. The other boundary condition is established by the assumption of radial symmetry, which implies that the radial velocity is zero at the center line and because $U_{i=1} = U_{i=-1}$, the a_0 term can be added to the c_0 term.

The first and last rows of the tri-diagonal matrix are then, respectively,

$$b_0 U_{0,j+1} + (c_0 + a_0)U_{1,j+1} = r_0 = r_0^* \qquad \text{(eqn. A8)}$$

and

$$a_{I-1}U_{I-2,j+1} + b_{I-1}U_{I-1,j+1} = r_{I-1} - c_{I-1}U_{I,j+1} = r_{I-1}^* \qquad \text{(eqn. A9)}$$

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